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Brown-Henneaux's canonical approach to topologically massive gravity

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Abstract: We analyze the symmetry realized asymptotically on the two dimensional boundary of AdS_3 geometry in topologically massive gravity, which consists of the gravitational Chern-Simons term as well as the usual Einstein-Hilbert and negative cosmological constant terms. Our analysis is based on the conventional canonical method and proceeds along the line completely parallel to the original Brown and Henneaux's. In spite of the presence of the gravitational Chern-Simons term, it is confirmed by the canonical method that the boundary theory actually has the conformal symmetry satisfying the left and right moving Virasoro algebras. The central charges of the Virasoro algebras are computed explicitly and are shown to be left-right asymmetric due to the gravitational Chern-Simons term. It is also argued that the Cardy's formula for the BTZ black hole entropy capturing *all* higher derivative corrections agrees with the extended version of the Wald's entropy formula. The M5-brane system is illustrated as an application of the present calculation.

Keywords: AdS-CFT Correspondence, Black Holes, Chern-Simons Theories, Black Holes in String Theory.

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Contents

1. Introduction

The three dimensional spacetime has been one of the interesting testing grounds to uncover quantum natures of gravity. Especially, the three dimensional gravity with negative cosmological constant has been paid much attention, since this system admits a globally AdS_3 geometry as a vacuum [1], and the black hole solution of Bañados, Teitelboim and Zanelli (BTZ) [2] as excited states. Moreover, this system can equivalently be analyzed by mapping to gauge Chern-Simons action [3].

One of the interesting properties of the AdS_3 geometry is that there exists an asymptotic symmetry at the boundary, described by two dimensional conformal field theory (CFT). By using a canonical formalism of the Einstein-Hilbert gravity, Brown and Henneaux [4] successfully constructed left- and right-moving Virasoro algebras at the boundary, which share a common nontrivial value for their central charges.

The existence of the two dimensional CFT is inferred if we embed this system in M-theory [5]. The low energy limit of the M-theory is well described by eleven dimensional supergravity, and after compactification on Calabi-Yau (CY) 3-fold, it becomes five dimensional supergravity [6]. An M5-brane which wraps on four cycles in CY_3 corresponds to a string-like black object in five dimensional supergravity, and after taking near horizon limit, the geometry becomes $\mathrm{AdS}_3 \times \mathrm{S}^2$. The AdS_3 geometry appears after the dimensional reduction of S^2 part. On the other hand, the field theory on the M5-brane is well described by two dimensional CFT after reducing four dimensional part which wraps on four cycles in CY_3. In this way we can understand $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence via the M5-brane wrapping on the CY_3 [7−9].

The three dimensional theory relevant to the M-theory includes both the gravitational Chern-Simons term and other matter fields containing higher derivative terms. Let us recall in this connection that Saida and Soda [10] have previously studied the higher derivatives without the Chern-Simons term. By applying frame transformation method [11], they mapped the higher derivative action to the canonical Einstein-Hilbert one. In the case of BTZ black hole, this frame transformation just scales the original metric, and it becomes possible to calculate the modification of the Virasoro central charges by the simple scaling argument. Both left and right central charges scale in the same way and agree with each other, even if the higher derivative terms are included.

In this paper, we generalize the work of ref. [10] by including the gravitational Chern-Simons term. This cannot be dealt with by the simple scaling argument, and we need to consider the canonical formalism in a conventional way. It has been argued by Gupta and Sen in ref. [12] that the method of field redefinition and consistent truncation transforms the three dimensional gravity theory into the one consisting of only three terms: the Einstein-Hilbert, the cosmological constant and the gravitational Chern-Simons terms. Such a three dimensional theory with negative cosmological constant is often referred to as topologically massive gravity (TMG) [13, 14]. The action is given by

$$\mathcal{S}_{\mathrm{TMG}} = \frac{1}{16\pi G_N}\int d^3x \left(\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{CS}}\right), \tag{1.1}$$

$$\mathcal{L}_{\mathrm{EH}} = \sqrt{-G}\left(R + \frac{2}{\ell^2}\right), \tag{1.2}$$

$$\mathcal{L}_{\mathrm{CS}} = \frac{\beta}{2}\sqrt{-G}\epsilon^{IJK}\left(\Gamma^P{}_{IQ}\partial_J\Gamma^Q{}_{KP} + \frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\right). \tag{1.3}$$

The cosmological constant $-2/\ell^2$ in $\mathcal{L}_{\mathrm{EH}}$ is negative and β is a coupling constant with the dimension of the length. The determinant of the three dimensional metric G_{IJ} is denoted by G, the three dimensional Christoffel symbol is by $\Gamma^P{}_{IQ}$ and the capital letters P, I, Q etc. label the three dimensional space-time indices, t, r and ϕ. Note also that ϵ^{MNO} is a covariantly constant tensor and $\sqrt{-G}\epsilon^{MNO}$ is just a constant.

Notice that (1.3) contains third derivative with respect to the time. The canonical formalism of such a system can be accomplished by using the Ostrogradsky method, where a new variable is introduced to reduce the number of the time derivative [15]. For the gravitational Chern-Simons term, it is convenient to employ the generalized version of it [16, 17]. Then it is possible to define the Hamiltonian, and from its variation we can extract the global charges, such as a mass, an angular momentum and central charges in

TMG. The asymptotic symmetry in TMG is again described by the left and right moving Virasoro algebras, whose central charges are, however, not the same as we will show later. The central charges has been derived in previous literatures in several ways [7, 8, 18, 19, 12].

The entropy of the black hole may be evaluated by using the Cardy's formula together with the modified values of the central charges. There is an important remark that the black hole entropy computed on the basis of the Cardy's formula should not be compared with the Wald's formula [20] in its original form which is applicable only for manifestly diffeomorphism invariant theories. The formula should be compared with the one given recently in refs. [19, 21], where a modification has been made so that one can include such a term as the Chern-Simons'. The agreement of both entropies are confirmed in our canonical formalism.

The structure of our work is as follows. In section 2 we present the modified version of the Wald's black hole entropy formula, paying a particular attention to higher derivative corrections including those of the Chern-Simons term. The framework of deriving the asymptotic symmetry is discussed in section 3. The calculation of the Virasoro central charges in TMG is given in section 4, in the Ostrogradsky method which is adapted to the cases of higher derivative terms. The mass and the angular momentum of the BTZ black hole is also discussed, including the effects due to the Chern-Simons term. In section 5, we discuss all of the higher derivative corrections and compare our black hole entropy formulas with the modified version of Wald's given in section 2. Our calculation is shown to be useful in the application of M5 system. Appendix A is devoted to a detailed discussion on the difference between gravitational and Lorentz Chern-Simons terms. Some of the calculational details are relegated to appendix B.

2. Most general entropy formula for BTZ black holes

Before starting to discuss the canonical method and the CFT approach to the BTZ black hole, we here concentrate on the macroscopic treatment of black hole entropy à la Wald (See refs. [22]). We pay a particular attention to, and try to include the effects of higher derivative terms together with the Chern-Simons' in as general a way as possible. For such a purpose we begin with the following Lagrangian

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left[f(R_{IJ},G_{IJ}) + \frac{2}{\ell_0^2}\right] + \frac{1}{16\pi G_N}\int d^3x\mathcal{L}_{\rm CS}. \tag{2.1}$$

Here f is a generally covariant part and is supposed to contain all possible higher derivatives. In three dimensional spacetime, the Riemann and the Ricci tensors have both six components and are related by the formula $R^{IJ}{}_{KL} = 4G^{(I}{}_{(K}R^{J)}{}_{L)} - RG^{(I}{}_{(K}G^{J)}{}_{L)}$. Therefore, f is assumed to be a functional of the Ricci tensor and the metric only. Note that the negative cosmological constant in (2.1) is denoted by $-2/\ell_0^2$.

The action (2.1) is diffeomorphism invariant up to the total derivatives. Due to the gravitational Chern-Simons term and other higher derivative terms, the Einstein equation is modified as

$$\frac{1}{2}G^{IJ}\left(f + \frac{2}{\ell_0^2}\right) + \frac{\partial f}{\partial G_{IJ}} + T^{IJ} = \beta\epsilon^{KL(I}\mathcal{D}_K R^{J)}_L. \tag{2.2}$$

Here we denote

$$T^{IJ} = \frac{1}{2}\left(\mathcal{D}_K\mathcal{D}^I P^{KJ} + \mathcal{D}_K\mathcal{D}^J P^{IK} - \Box P^{IJ} - G^{IJ}\mathcal{D}_K\mathcal{D}_L P^{KL}\right), \quad P^{IJ} = \frac{\partial f}{\partial R_{IJ}}. \tag{2.3}$$

$\mathcal{D}_I$ is the usual covariant derivative.

Various solutions to (2.2) would be possible, but it is known that the AdS_3 geometry satisfying

$$\frac{1}{2}G^{IJ}\left(R + \frac{2}{\ell^2}\right) - R^{IJ} = 0 \tag{2.4}$$

with some constant ℓ is certainly a solution to (2.2). This can be seen by the following argument. If (2.4) is satisfied, then the scalar curvature is just a constant ($R = -6/\ell^2$) and the metric and the Ricci tensors are proportional ($R_{IJ} = -2G_{IJ}/\ell^2$). We can see that (2.3) and the right-hand-side of (2.2) vanish. Eq. (2.2) turns out to be a relation that fixes ℓ as a function of ℓ_0. This may be regarded as an "effective" cosmological constant due to the higher derivative terms.[1]

The vacuum solution to (2.4) is the global AdS geometry with the radius ℓ

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{2.5}$$

As an excited state the BTZ black hole solution [2]

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2, \\ N^2 &= \left(\frac{r}{\ell}\right)^2 + \left(\frac{4G_N j}{r}\right)^2 - 8G_N m\,, \qquad N^\phi = \frac{4G_N j}{r^2}, \end{aligned} \tag{2.6}$$

is also allowed, which preserves the local AdS_3 symmetry and is constructed by global identification of independent points on (2.5). In the Einstein-Hilbert gravity with negative cosmological constant, parameters m and j correspond to the mass and angular momentum of the BTZ black hole. In general, however, the definition of the mass and angular momentum must be changed by taking into account of other higher derivative terms. As we will see explicitly in section 4 in the canonical formalism, the effective mass M and the effective angular momentum J of the BTZ black hole (2.6) are represented by linear combinations of m and j when the Chern-Simons term is present.

It is well known that the Bekenstein-Hawking's area law for the black hole entropy is modified by Wald's entropy formula for general covariant theories which include higher derivative terms [20]. In those theories, however, which are not manifestly invariant under the diffeomorphism, that formula cannot be applied directly and must be modified. The extended Noether method including the contribution of the non-covariant terms was discussed in [21]. In general, the non-covariant terms such as the gravitational Chern-Simons term, which is one of the higher derivative terms, modify the Noether charge in a slightly different fashion from Wald's formula. As a result, the black hole entropy receives the higher derivative correction further. In practice we have to determine the corrections to

[1]Throughout this paper, the effective cosmological constant $-2/\ell^2$ always appears in the solutions.

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the Wald's entropy formula on a case-by-case basis. For the gravitational Chern-Simons term in three dimensions the additional correction ΔS to the entropy is found to be

$$\Delta S = \frac{\beta}{4G_N} \int_H \varepsilon^J{}_I \Gamma^I{}_{JK} dx^K. \tag{2.7}$$

Here ε^{IJ} is a binormal vector on the horizon H. The equivalent results for the three dimensional gravitational Chern-Simons term were obtained by several others [19, 18].

With the help of this, the full entropy of the BTZ black hole are, therefore, calculated as follows:

$$\begin{aligned} S &= -\frac{1}{8G_N} \oint_{r_+} d\phi \sqrt{G_{\phi\phi}} \frac{\partial f}{\partial R_{IK}} G^{JL} \varepsilon_{IJ} \varepsilon_{KL} + \frac{\beta}{4G_N} \oint_{r_+} d\phi \, \varepsilon^{JI} \Gamma_{IJ\phi} \\ &= \frac{1}{4G_N} \Omega \oint_{r_+} d\phi \sqrt{G_{\phi\phi}} + \frac{\beta}{4G_N} \oint_{r_+} d\phi \frac{r_+ r_-}{\ell r} \\ &= \frac{\pi \Omega}{2G_N} r_+ + \frac{\pi \beta}{2G_N \ell} r_-, \end{aligned} \tag{2.8}$$

where the conformal factor Ω is defined by

$$\Omega = \frac{1}{3} G_{IJ} \frac{\partial f}{\partial R_{IJ}}. \tag{2.9}$$

This Ω is just a constant for the BTZ black hole solution (2.6), and $\Omega = 1$ for the Einstein-Hilbert action. By substituting explicit values of $r_\pm = \sqrt{2G_N \ell(\ell m + j)} \pm \sqrt{2G_N \ell(\ell m - j)}$ into (2.8), we finally obtain the entropy formula

$$S = \frac{\pi}{2G_N} \left\{ \left(\Omega + \frac{\beta}{\ell}\right) \sqrt{2G_N \ell^2 \left(m + \frac{j}{\ell}\right)} + \left(\Omega - \frac{\beta}{\ell}\right) \sqrt{2G_N \ell^2 \left(m - \frac{j}{\ell}\right)} \right\}. \tag{2.10}$$

This is the macroscopic entropy including *all* higher derivative corrections in three dimensions. In the present paper we consider only the parameter region of $\Omega\ell > \beta > 0$ just for simplicity. The situation where $\Omega < 0$ or $\beta > \Omega\ell$ has been discussed in [22]. In the following sections, by generalizing the original Brown-Henneaux's canonical approach, we shall show that the expression (2.10) is in perfect agreement with the Cardy's formula for the CFT on the two dimensional boundary.

3. Hamiltonian formalism and Virasoro algebras

As long as the BTZ black holes are concerned, our analyses of the asymptotic symmetry associated with (1.1) will go along a line quite parallel to those in ref. [4] where only the Einstein-Hilbert action is considered. It is therefore convenient to summarize key ingredients of Brown and Henneaux's work which are not altered when we take the Chern-Simons term into our consideration.

First of all let us specify the boundary conditions so that field configurations behave as "asymptotically AdS_3". We require that the metric should behave at the spatial infinity

$r \to \infty$ as

$$
\begin{aligned}
G_{tt} &= -\frac{r^2}{\ell^2} + \mathcal{O}(1)\,, & G_{tr} &= \mathcal{O}(r^{-3})\,, & G_{t\phi} &= \mathcal{O}(1), \\
G_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4})\,, & G_{r\phi} &= \mathcal{O}(r^{-3})\,, & G_{\phi\phi} &= r^2 + \mathcal{O}(1),
\end{aligned} \tag{3.1}
$$

which is in accordance with the behavior in (2.5) and (2.6). The vector fields $(\bar{\xi}^0, \bar{\xi}^r, \bar{\xi}^\phi)$ that transform the metric while preserving the boundary conditions (3.1) are not strongly restricted but are allowed to be a general class of functions. In fact, by using the coordinates of $x^\pm = \frac{t}{\ell} \pm \phi$, the n-th Fourier component of the vector fields is given by

$$
\bar{\xi}^t = \frac{\ell}{2} e^{inx^\pm} \left(1 - \frac{\ell^2 n^2}{2r^2}\right), \quad \bar{\xi}^r = -i\frac{nr}{2} e^{inx^\pm}, \quad \bar{\xi}^\phi = \pm\frac{1}{2} e^{inx^\pm} \left(1 + \frac{\ell^2 n^2}{2r^2}\right). \tag{3.2}
$$

In this paper we call the above vector fields “Killing vectors”. For later use, we assign explicit notations for these Killing vectors:

$$
\xi_n^\pm \equiv \bar{\xi}^I \partial_I = e^{inx^\pm} \left(\partial_\pm - \frac{\ell^2 n^2}{2r^2} \partial_\mp - \frac{inr}{2} \partial_r\right), \tag{3.3}
$$

where $\partial_\pm = \frac{1}{2}(\ell \partial_t \pm \partial_\phi)$. The algebraic structure of the symmetry is encoded in the Killing vector and in fact we can directly compute the commutation relations of these differential operators

$$
\left[\xi_m^\pm, \xi_n^\pm\right] = -i\,(m-n)\,\xi_{m+n}^\pm, \qquad \left[\xi_m^+, \xi_n^-\right] = \mathcal{O}(r^{-4}). \tag{3.4}
$$

This result clearly shows that the asymptotically AdS_3 spacetime is endowed with the two dimensional conformal symmetry.

In order to evaluate the central extension of the Virasoro algebras, we have to know the asymptotic behaviors of the canonical variables and we introduce the $(2+1)$-dimensional decomposition of the three dimensional metric G_{IJ} as

$$
G_{IJ} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & g_{ij} \end{pmatrix}. \tag{3.5}
$$

Here $g_{ij}, (i, j = r, \phi)$ is the two dimensional metric. The lapse and shift functions are denoted by N and N_i, respectively. The Einstein-Hilbert action is rewritten as usual by

$$
\mathcal{L}_{\mathrm{EH}} = \sqrt{g} N \left(r + \frac{2}{\ell^2} + K^{ij} K_{ij} - K^2\right), \tag{3.6}
$$

where r is the scalar curvature made out of g_{ij}, and

$$
K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right), \tag{3.7}
$$

$$
K = g^{ij} K_{ij}. \tag{3.8}
$$

The dot over g_{ij} means the t-derivative and $\mathcal{D}_i$ is the covariant derivative with respect to g_{ij}. The momentum variable π^{ij} conjugate to g_{ij} is given by $\pi^{ij} = \sqrt{g}(K^{ij} - g^{ij} K)$ for the case of Einstein-Hilbert action, and the Hamiltonian $\mathcal{H}$ is the Legendre transform of $\mathcal{L}_{\rm EH}$, i.e., $\mathcal{H} = \pi^{ij}\dot{g}_{ij} - \mathcal{L}_{\rm EH}$.

The Hamiltonian consists of the usual combination of the constraints together with appropriate surface term $Q[\xi]$,

$$H[\xi] = \int d^2x \left(\xi^0\mathcal{H} + \xi^i\mathcal{H}_i\right) + Q[\xi]. \tag{3.9}$$

The added term $Q[\xi]$ must be determined so that it cancels the surface terms produced by the first term in (3.9) under field variation and is a generator of the possible surface deformation [23]. The vector fields (ξ^0, ξ^r, ξ^ϕ) denote such allowed surface deformation and are related to the spacetime vector $(\bar{\xi}^0, \bar{\xi}^r, \bar{\xi}^\phi)$ via

$$(\xi^0, \xi^r, \xi^\phi) = (N\bar{\xi}^t, \bar{\xi}^r + N^r\bar{\xi}^t, \bar{\xi}^\phi + N^\phi\bar{\xi}^t). \tag{3.10}$$

The asymptotic behaviors (3.1) are now translated into those of the canonical variables as

$$g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}), \qquad g_{r\phi} = \mathcal{O}(r^{-3}), \qquad g_{\phi\phi} = r^2 + \mathcal{O}(1), \tag{3.11}$$

$$N = \frac{r}{\ell} + \mathcal{O}(r^{-1}), \qquad N^r = \mathcal{O}(r^{-1}), \qquad N^\phi = \mathcal{O}(r^{-2}). \tag{3.12}$$

The behaviors of the canonical conjugate variables are also derived with the help of (3.7), (3.11) and (3.12):

$$\pi^{rr} = \mathcal{O}(r^{-1}) \quad , \quad \pi^{r\phi} = \mathcal{O}(r^{-2}) \quad , \quad \pi^{\phi\phi} = \mathcal{O}(r^{-5}). \tag{3.13}$$

It has been known that conditions (3.11), (3.12) and (3.13) are preserved under the Hamiltonian evolution provided that we impose the Hamiltonian constraints. The generator $Q[\xi]$ in (3.9) is found by taking into account the asymptotic behaviors of the canonical variables up to a constant term, which is adjusted so that the charge $Q[\xi]$ vanishes for the globally AdS space.

The algebraic structure of symmetric transformation group is given by the Poisson bracket algebra of Hamiltonian generator $H[\xi]$:

$$\{H[\xi], H[\eta]\}_{\rm P} = H\big[[\xi,\eta]\big] + K[\xi,\eta], \tag{3.14}$$

where $K[\xi,\eta]$ is the possible central extension. The central charge may be evaluated by noting that the Dirac bracket $\{Q[\xi], Q[\eta]\}_{\rm D}$ is the change of $Q[\xi]$ under the surface deformation due to $Q[\eta]$, i.e., $\delta_\eta Q[\xi] = \{Q[\xi], Q[\eta]\}_{\rm D}$. The charge $Q[\xi]$ forms a conformal group with a central extension $\{Q[\xi], Q[\eta]\}_{\rm D} = Q\big[[\xi,\eta]\big] + K[\xi,\eta]$, and we immediately get $\delta_\eta Q[\xi] = Q\big[[\xi,\eta]\big] + K[\xi,\eta]$. Since $Q\big[[\xi,\eta]\big] = 0$ if we set the initial condition so that the charge vanishes for a globally AdS space, the evaluation of the central charge reduces to

$$K\big[\xi,\eta\big] = \delta_\eta Q[\xi]. \tag{3.15}$$

In the case of Einstein-Hilbert action, the explicit form is given by

$$\delta_\eta Q[\xi] = \int d\phi \left[\sqrt{g}S^{ijkr}\left\{\xi^0\mathcal{D}_k\delta_\eta g_{ij} - \mathcal{D}_k\xi^0\delta_\eta g_{ij}\right\} + 2\xi^i\pi^{jr}\delta_\eta g_{ij} + 2\xi_i\delta_\eta\pi^{ir} - \xi^r\pi^{ij}\delta_\eta g_{ij}\right], \tag{3.16}$$

where S^{ijkl} is defined by

$$S^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right). \tag{3.17}$$

(Derivation of the above equations will be explained in the case of TMG in section 4.)

Putting the Killing vectors (3.3) for ξ, we define the Virasoro generators by $\hat{L}_n^{\pm} = Q[\xi_n^{\pm}]$. Replacing the Dirac brackets by a commutator ($\{,\}_{\mathrm{D}} \to -i[,]$), the commutation relations become

$$\begin{aligned}
[\hat{L}_m^+, \hat{L}_n^+] &= (m-n)\hat{L}_{m+n}^+ + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0}, \\
[\hat{L}_m^-, \hat{L}_n^-] &= (m-n)\hat{L}_{m+n}^- + \frac{c_R}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m^+, L_n^-] &= 0,
\end{aligned} \tag{3.18}$$

and the central charges have been calculated in [4] as

$$c_L = c_R = \frac{3\ell}{2G_N}. \tag{3.19}$$

Once we get the central charges, it is straightforward to obtain the BTZ black hole entropy by using the Cardy's formula

$$\begin{aligned}
S &= 2\pi\sqrt{\frac{1}{6}c_L L_0^+} + 2\pi\sqrt{\frac{1}{6}c_R L_0^-} \\
&= \frac{\pi}{2G_N}\sqrt{2G_N\ell^2\left(m + \frac{j}{\ell}\right)} + \frac{\pi}{2G_N}\sqrt{2G_N\ell^2\left(m - \frac{j}{\ell}\right)}.
\end{aligned} \tag{3.20}$$

Here $L_0^{\pm}$ are the eigenvalues of $\hat{L}_0^{\pm}$ and are related to the mass m and angular momentum j of the black hole by the formulae $L_0^+ + L_0^- = m\ell$ and $L_0^+ - L_0^- = j$.

In section 2 we have seen that even in the presence of higher derivative interactions, (2.5) and (2.6) are still solutions to the equations of motion with the effective cosmological constant $-2/\ell^2$. There occurs a phenomenon of rescaling of the charges due to the higher derivative terms, and in fact the first term in the entropy formula (2.8) is rescaled by the factor Ω defined by (2.9). The question that we would like to address ourselves here is what this rescaling phenomenon looks like in the CFT framework [10].

Let us start with the diffeomorphism invariant Lagrangian without the gravitational Chern-Simons term,

$$\mathcal{L} = \sqrt{-G}\left[f(R_{IJ}, G_{IJ}) + \frac{2}{\ell_0^2}\right]. \tag{3.21}$$

Higher derivative terms are again included in f. The important point is that the Lagrangian constructed out of the metric and the Ricci tensor is equivalent to the Einstein-Hilbert Lagrangian with matter fields after the frame transformation [11]. The metric $\tilde{G}^{IJ}$ in the Einstein frame is defined to be

$$\tilde{G}^{IJ} = \left|\det\left(\frac{\partial\mathcal{L}}{\partial R_{KL}}\right)\right|^{-1}\frac{\partial\mathcal{L}}{\partial R_{IJ}}, \tag{3.22}$$

and when BTZ solution (2.6) is substituted into the above expression, we obtain

$$\tilde{G}_{IJ} = \Omega^2 G_{IJ}. \tag{3.23}$$

The conformal factor Ω is already defined in (2.9) and becomes a constant here. This means that canonical variables are scaled like $\tilde{g}_{ij} = \Omega^2 g_{ij}$, $\tilde{N} = \Omega N$, $\tilde{N}^i = N^i$ and $\tilde{\pi}^{ij} = \Omega^{-1}\pi^{ij}$. From these scaling rules, we find that the mass M, the angular momentum J and the central charges c_L and c_R, which are evaluated by using (3.16), are multiplied by Ω as

$$M = \Omega m, \qquad J = \Omega j, \qquad c_L = c_R = \Omega \frac{3\ell}{2G_N}. \tag{3.24}$$

Then the eigenvalues of the Virasoro generators are also linearly scaled as $L_0^{\pm} = \frac{1}{2}(M\ell \pm J) = \frac{1}{2}\Omega(m\ell \pm j)$, and these considerations lead us to conclude that effects of higher derivative terms to the BTZ black hole are all summarized by the rescaling i.e.,

$$S = \frac{\pi}{2G_N}\Omega\sqrt{2G_N\ell^2\left(m + \frac{j}{\ell}\right)} + \frac{\pi}{2G_N}\Omega\sqrt{2G_N\ell^2\left(m - \frac{j}{\ell}\right)}. \tag{3.25}$$

Of course this agrees with (2.10) with $\beta = 0$ [10]. We will come to this scaling rule again in section 5 after establishing the canonical formalism and CFT description of the Chern-Simons term.

4. Generalization to topologically massive gravity

4.1 Canonical formalism of charges in TMG

In this section we investigate the canonical formalism of TMG with negative cosmological constant, and derive the expression of global charges in this system. From these global charges, we will confirm that the mass and the angular momentum of the BTZ black hole and the central charges of the boundary CFT are all modified in TMG.

Let us apply the ADM decomposition to the Lagrangian of the topologically massive gravity. The action of TMG is given by (1.1), and the Einstein-Hilbert term with negative cosmological constant is decomposed as in (3.6). After straightforward but tedious calculation, the gravitational Chern-Simons term can be decomposed up to total derivative terms into [24, 17]

$$\begin{aligned}
&\sqrt{-G}\epsilon^{IJK}\left(\Gamma^P{}_{IQ}\partial_J\Gamma^Q{}_{KP} + \frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\right)\\
&\quad\cong \sqrt{g}\epsilon^{mn}\Big\{2\dot{K}_{mk}K_n{}^k + \dot{\gamma}^x{}_{my}\gamma^y{}_{nx} - 2\partial_k N\mathcal{D}_n K_m{}^k + 2\mathcal{D}_n\partial_k N K_m{}^k\\
&\qquad\qquad + NK_y{}^x\partial_m\gamma^y{}_{nx} - \partial_m N K_y{}^x\gamma^y{}_{nx} - N\mathcal{D}_m K_y{}^x\gamma^y{}_{nx}\\
&\qquad\qquad - 2N^iK_i{}^l\mathcal{D}_nK_{ml} + 2\mathcal{D}_nN^iK_i{}^lK_{ml} + 2N^i\mathcal{D}_nK_i{}^lK_{ml}\\
&\qquad\qquad + 2\mathcal{D}_kN^iK_{ni}K_m{}^k + \mathcal{D}_jN^i\partial_m\gamma^j{}_{ni} - \mathcal{D}_m\mathcal{D}_jN^i\gamma^j{}_{ni}\Big\}\\
&\quad\cong 2\sqrt{g}\epsilon^{mn}\dot{K}_{mk}K_n{}^k + \sqrt{g}N\big\{4\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k - 2A^{kl}K_{kl}\big\}\\
&\qquad + \sqrt{g}N^i\big\{-4\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml} - 2\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) + \epsilon_{ij}\partial^j r + 2\mathcal{D}_kA_i{}^k\big\}.
\end{aligned} \tag{4.1}$$

Here ϵ^{mn} is a covariantly constant antisymmetric tensor in two dimensions, and $\mathcal{D}_i$ is the covariant derivative. In the above, we used that the Riemann tensor in two dimensions is expressed by the scalar curvature as $r^i{}_{jmn} = \frac{1}{2}(\delta^i_m g_{jn} - \delta^i_n g_{jm})r$. A^{ij} is a symmetric tensor which is defined by the following equation.

$$\int d^3x \sqrt{g}\epsilon^{mp}\dot{\gamma}^l{}_{mn}\gamma^n{}_{pl} = -\int d^3x \sqrt{g} A^{ij}\dot{g}_{ij}. \tag{4.2}$$

The dot is used to represent time derivative ∂_t. By defining

$$T^{ijk}_{mno} \equiv \frac{1}{2}(\delta^k_m \delta^{(i}_o \delta^{j)}_n + \delta^k_n \delta^{(i}_o \delta^{j)}_m - \delta^k_o \delta^{(i}_m \delta^{j)}_n), \tag{4.3}$$

the time derivative of the affine connection is expressed as $\dot{\gamma}^l{}_{mn} = g^{lo} T^{ijk}_{mno}\mathcal{D}_k \dot{g}_{ij}$. After the partial integration in (4.2), A^{ij} is explicitly written as

$$\begin{aligned} A^{ij} &= \epsilon^{mp} g^{lo} T^{ijk}_{mno} \mathcal{D}_k \gamma^n{}_{pl} \\ &= \frac{1}{4}\epsilon^{kl}\mathcal{D}_k \gamma^i{}_l{}^j + \frac{1}{4}\epsilon^{il}\mathcal{D}_k \gamma^k{}_l{}^j - \frac{1}{4}\epsilon^{il}\mathcal{D}_k \gamma^j{}_l{}^k + (i \leftrightarrow j). \end{aligned} \tag{4.4}$$

Since A^{ij} depends on the affine connection in an explicit way, it does not behave as a tensor. The derivation of (4.1) is explained in appendix A by focusing on the difference from Lorentz Chern-Simons term.[2]

The explicit form of the extrinsic curvature is given by (3.7). Then (4.1) contains third derivatives with respect to time. It is known that the canonical formalism of such system is done by using Ostrogradsky method [15] in which Lagrange multiplier is introduced. For instance, if there is a Lagrangian $\mathcal{L}(g, \dot{g}, \ddot{g})$, then we define $\mathcal{L}^*(g, \dot{g}, h, \dot{h}, v) = \mathcal{L}(g, h, \dot{h}) + v(\dot{g} - h)$ and construct the Hamiltonian in the usual way. In the case of TMG , it is useful to apply modified version of Ostrogradsky method as discussed in ref. [16, 17]. In the modified Ostrogradsky method, the extrinsic curvature is dealt with an independent variable. At the same time, Lagrange multiplier should be introduced to give a proper constraint. Following this prescription, the Lagrangian of the TMG is given by

$$\begin{aligned} \mathcal{L}_{\text{TMG}} &= \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CS}} \\ &= \sqrt{g}N\left(r + \frac{2}{\ell^2} + K^{ij}K_{ij} - K^2\right) + v^{ij}(\dot{g}_{ij} - 2NK_{ij} - 2\mathcal{D}_i N_j) \\ &\quad + \beta\sqrt{g}\epsilon^{mn}\dot{K}_{mk}K_n{}^k + \beta\sqrt{g}N\left(2\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k - A^{kl}K_{kl}\right) \\ &\quad + \beta\sqrt{g}N^i\left\{-2\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} - \epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) + \frac{1}{2}\epsilon_{ij}\partial^j r + \mathcal{D}_k A_i{}^k\right\}. \end{aligned} \tag{4.5}$$

Canonical variables in this Lagrangian are g_{ij}, $\dot{g}_{ij}$, K_{ij} and $\dot{K}_{ij}$, and N, N^i and v^{ij} are Lagrange multipliers. Note that v_{ij}, which is not a tensor, is symmetric under the exchange of indices.

[2]The notations employed here are slightly different from those in ref. [24, 17].

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By using this Lagrangian, we can construct the Hamiltonian in the canonical procedure. As usual, momenta conjugate to $\dot{g}_{ij}$ and $\dot{K}_{ij}$ are defined as

$$\pi^{ij} \equiv \frac{\delta \mathcal{L}_{\mathrm{TMG}}}{\delta \dot{g}_{ij}} = v^{ij},$$
$$\Pi^{ij} \equiv \frac{\delta \mathcal{L}_{\mathrm{TMG}}}{\delta \dot{K}_{ij}} = \beta \sqrt{g} \epsilon^{ik} K_k{}^j. \tag{4.6}$$

Note that not π^{ij} and Π^{ij} but $g^{-\frac{1}{2}}\pi^{ij}$ and $g^{-\frac{1}{2}}\Pi^{ij}$ do behave like tensors. From the second equation, we see that Π_{ij} and K_{ij} are not independent and the system is constrained. Again, such a kind of the constraint should be taken into account by introducing Lagrange multiplier in the Hamiltonian formalism. Up to total derivative terms, the Hamiltonian of TMG is expressed as

$$\begin{aligned}
&\mathcal{H}_{\mathrm{TMG}} \\
&\quad = \pi^{ij}\dot{g}_{ij} + \Pi^{ij}\dot{K}_{ij} - \mathcal{L}_{\mathrm{TMG}} + f_{ij}\big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\big) \\
&\quad \cong \sqrt{g}N\bigg\{ -r - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k + \big(2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl}\big)K_{kl}\bigg\} \\
&\qquad + \sqrt{g}N^i\bigg\{2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^j r - \mathcal{D}_j\big(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\big)\bigg\} \\
&\qquad + f_{ij}\big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\big),
\end{aligned} \tag{4.7}$$

where f_{ij} is the Lagrange multiplier. The validity of this Hamiltonian will be confirmed by explicitly deriving the equations of motion in TMG. In fact we will show below that a part of the equations of motion matches with the one obtained from the Lagrangian formalism in three dimensions.

Let us consider the variation of the Hamiltonian by fluctuating g_{ij}, π^{ij}, K_{ij}, Π^{ij}, N, N^i and f_{ij}. The variation of the Hamiltonian contains total derivative terms. Though those terms are very important to define charges, we neglect them for a while to make the argument as simple as possible. Then up to total derivative terms, the variation of the Hamiltonian under the fluctuations of N, N^i and f_{ij} is calculated as

$$\begin{aligned}
&\delta_{(N,N^i,f_{ij})}\mathcal{H}_{\mathrm{TMG}} \\
&\quad = \delta N\sqrt{g}\bigg\{ -r - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k + \big(2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl}\big)K_{kl}\bigg\} \\
&\qquad + \delta N^i\sqrt{g}\bigg\{2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^j r - \mathcal{D}_j\big(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\big)\bigg\} \\
&\qquad + \delta f_{ij}\bigg\{\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\bigg\}.
\end{aligned} \tag{4.8}$$

From this we see that the canonical variables are constrained like

$$-r - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k + (2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl})K_{kl} = 0, \tag{4.9}$$

$$2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^j r - \mathcal{D}_j\big(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\big) = 0, \tag{4.10}$$

$$\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j = 0. \tag{4.11}$$

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Note that neither $g^{-\frac{1}{2}}\pi^{ij}$ nor A^{ij} behaves like tensors. The linear combination of $(2g^{-\frac{1}{2}}\pi^{ij}+\beta A^{ij})$, however, does behave as a tensor.

Next, up to total derivative terms, the variation of the Hamiltonian under the fluctuations of g_{ij}, K_{ij}, π^{ij} and Π^{ij} is calculated as

$$\begin{aligned}
&\delta_{(g_{ij},K_{ij},\pi^{ij},\Pi^{ij})}\mathcal{H}_{\mathrm{TMG}} \\
&= \delta\pi^{ij}\Big\{2NK_{ij}+2\mathcal{D}_iN_j\Big\}+\delta\Pi^{ij}\Big\{f_{ij}\Big\}+\delta(\sqrt{g}\beta A^{ij})\Big\{NK_{ij}+\mathcal{D}_iN_j\Big\} \\
&\quad+\delta g_{ij}\sqrt{g}\Big\{N\Big(r^{ij}-\frac{1}{2}g^{ij}r-\frac{1}{\ell^2}g^{ij}\Big)+2N(K^{ik}K^j{}_k-KK^{ij})-\frac{1}{2}Ng^{ij}(K^{kl}K_{kl}-K^2) \\
&\qquad-(\mathcal{D}^i\mathcal{D}^jN-g^{ij}\mathcal{D}_k\mathcal{D}^kN)+(2g^{-\frac{1}{2}}\pi^{k(i}+\beta A^{k(i})\mathcal{D}_kN^{j)})-\frac{1}{2}\mathcal{D}_k\big(N^k(2g^{-\frac{1}{2}}\pi^{ij}+\beta A^{ij})\big) \\
&\qquad+2\beta\epsilon^{mn}N\mathcal{D}^i\mathcal{D}_nK_m{}^j-2\beta\epsilon^{mn}N^kK_k{}^i\mathcal{D}_nK_m{}^j+2\beta\epsilon^{mn}g^{kl}T^{ijz}_{npl}\mathcal{D}_z(N^oK_o{}^pK_{mk}) \\
&\qquad-2\beta\epsilon^{mn}g^{kl}g^{op}\mathcal{D}_z\big(-\mathcal{D}_kNK_{mo}T^{ijz}_{nlp}+N\mathcal{D}_oK_{ml}T^{ijz}_{knp}+2N\mathcal{D}_nK_{o(l}T^{ijz}_{m)kp}\big) \\
&\qquad-\beta\epsilon^{mn}N^k\mathcal{D}^i(K_{nk}K_m{}^j)+2\beta\epsilon^{mn}g^{qk}g^{lo}\mathcal{D}_z\big(N^pK_{mk}K_{l(p}T^{ijz}_{n)qo}+N^pK_{np}K_{l(k}T^{ijz}_{m)qo}\big) \\
&\qquad-\frac{1}{2}\beta\epsilon^{mn}\delta^i_mN^j\partial_nr-\frac{1}{2}\beta\epsilon^{mn}\mathcal{D}_nN_mr^{ij}+\frac{1}{2}\beta\epsilon^{mn}S^{ijkl}\mathcal{D}_k\mathcal{D}_l\mathcal{D}_nN_m\Big\} \qquad (4.12)\\
&\quad+\delta K_{ij}\sqrt{g}\Big\{-2NK^{ij}+2Ng^{ij}K-2\beta\epsilon^{ik}\mathcal{D}_k\mathcal{D}^jN+N\big(2g^{-\frac{1}{2}}\pi^{ij}+\beta A^{ij}\big)+2\beta\epsilon^{mn}N^i\mathcal{D}_nK_m{}^j \\
&\qquad\qquad-2\beta\epsilon^{ik}\mathcal{D}_k(N^lK_l{}^j)+\beta\epsilon^{ik}\mathcal{D}_lN^jK_k{}^l-\beta\epsilon^{ik}\mathcal{D}^jN^lK_{lk}+\beta\epsilon^{ik}f_k{}^j\Big\}.
\end{aligned}$$

Here S^{ijkl} has been defined by (3.17). Note that A^{kl} is a function of g_{ij} and δA^{kl} depends linearly on δg_{ij}. From this, equations of motion for canonical variables are written as

$$\dot{g}_{ij}=2NK_{ij}+2\mathcal{D}_{(i}N_{j)}, \qquad (4.13)$$

$$\dot{K}_{ij}=f_{ij}, \qquad (4.14)$$

$$\begin{aligned}
\dot{\pi}^{ij}&=-\sqrt{g}\Big\{N\Big(r^{ij}-\frac{1}{2}g^{ij}r-\frac{1}{\ell^2}g^{ij}\Big)+2N(K^{ik}K^j{}_k-KK^{ij})-\frac{1}{2}Ng^{ij}(K^{kl}K_{kl}-K^2) \\
&\quad-(\mathcal{D}^i\mathcal{D}^jN-g^{ij}\mathcal{D}_k\mathcal{D}^kN)+(2g^{-\frac{1}{2}}\pi^{k(i}+\beta A^{k(i})\mathcal{D}_kN^{j)})-\frac{1}{2}\mathcal{D}_k\big(N^k(2g^{-\frac{1}{2}}\pi^{ij}+\beta A^{ij})\big) \\
&\quad+2\beta\epsilon^{mn}N\mathcal{D}^i\mathcal{D}_nK_m{}^j-2\beta\epsilon^{mn}N^kK_k{}^i\mathcal{D}_nK_m{}^j+2\beta\epsilon^{mn}g^{kl}T^{ijz}_{npl}\mathcal{D}_z(N^oK_o{}^pK_{mk}) \\
&\quad-2\beta\epsilon^{mn}g^{kl}g^{op}\mathcal{D}_z\big(-\mathcal{D}_kNK_{mo}T^{ijz}_{nlp}+N\mathcal{D}_oK_{ml}T^{ijz}_{knp}+2N\mathcal{D}_nK_{o(l}T^{ijz}_{m)kp}\big) \\
&\quad-\beta\epsilon^{mn}N^k\mathcal{D}^i(K_{nk}K_m{}^j)+2\beta\epsilon^{mn}g^{qk}g^{lo}\mathcal{D}_z\big(N^pK_{mk}K_{l(p}T^{ijz}_{n)qo}+N^pK_{np}K_{l(k}T^{ijz}_{m)qo}\big) \\
&\quad-\frac{1}{2}\beta\epsilon^{mn}\delta^i_mN^j\partial_nr-\frac{1}{2}\beta e\epsilon^{mn}\mathcal{D}_nN_mr^{ij}+\frac{1}{2}\beta\epsilon^{mn}S^{ijkl}\mathcal{D}_k\mathcal{D}_l\mathcal{D}_nN_m \qquad (4.15)\\
&\quad-\frac{1}{2}\beta\epsilon^{mn}T^{xyz}_{mlo}\Big(\dot{g}_{xy}g^{oi}g^{pj}\mathcal{D}_z\gamma^l{}_{np}+g^{op}g^{lr}\dot{g}_{xy}\mathcal{D}_k\gamma^q{}_{np}T^{ijk}_{zqr}-2g^{op}g^{qr}\dot{g}_{xy}\mathcal{D}_k\gamma^l{}_{q(p}T^{ijk}_{n)zr} \\
&\qquad-g^{op}g^{lq}\mathcal{D}_k\mathcal{D}_z\dot{g}_{xy}T^{ijk}_{npq}+g^{op}g^{lr}\mathcal{D}_k\dot{g}_{xy}\gamma^q{}_{np}T^{ijk}_{zqr}-2g^{op}g^{qr}\mathcal{D}_k\dot{g}_{xy}\gamma^l{}_{q(p}T^{ijk}_{n)zr}\Big)\Big\},
\end{aligned}$$

$$\begin{aligned}
\dot{\Pi}^{ij}&=-\sqrt{g}\Big\{-2NK^{ij}+2Ng^{ij}K-2\beta\epsilon^{ik}\mathcal{D}_k\mathcal{D}^jN+N\big(2g^{-\frac{1}{2}}\pi^{ij}+\beta A^{ij}\big)+2\beta\epsilon^{mn}N^i\mathcal{D}_nK_m{}^j \\
&\qquad-2\beta\epsilon^{ik}\mathcal{D}_k(N^lK_l{}^j)+\beta\epsilon^{ik}\mathcal{D}_lN^jK_k{}^l-\beta\epsilon^{ik}\mathcal{D}^jN^lK_{lk}+\beta\epsilon^{ik}f_k{}^j\Big\}. \qquad (4.16)
\end{aligned}$$

Note that (4.13) is used in (4.15).

Let us confirm that a part of equations of motion matches with the equation

$$E_{AB} \equiv R_{AB} - \frac{1}{2}\eta_{AB}R - \frac{1}{\ell^2}\eta_{AB} + \beta D_C R_{D(A}\epsilon_{B)}{}^{CD} = 0, \tag{4.17}$$

which is directly derived by the Lagrangian formalism in three dimensions. Note that the indices $A, B, \ldots$ are used for three dimensional local Lorentz frame. The covariant derivative D_C, which is defined by a spin connection, acts on the local Lorentz indices. By contracting (4.16) with K_{ij} and using (4.9) and (4.11), it is possible to derive

$$\begin{aligned}
0 &= -\beta\epsilon^{ik}K_{ij}(g^{jl}\dot{K}_{kl}) + 2NK_{ij}K^{ij} - 2NK^2 + 2\beta\epsilon^{ik}K_{ij}\mathcal{D}_k\mathcal{D}^j N \\
&\quad - N\left(r + \frac{2}{\ell^2} + K^{kl}K_{kl} - K^2 + 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k\right) \\
&\quad - 2\beta\epsilon^{mn}K_{ij}N^i\mathcal{D}_n K_m{}^j + 2\beta\epsilon^{ik}K_{ij}\mathcal{D}_k(N^l K_l{}^j) - \beta\epsilon^{ik}K_{ij}\mathcal{D}_l N^j K_k{}^l \\
&\quad + \beta\epsilon^{ik}K_{ij}\mathcal{D}^j N^l K_{lk} - \beta\epsilon^{ik}K_{ij}g^{jl}\dot{K}_{kl} \\
&= -N\left(r + \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2\right) + 2\beta\epsilon^{ij}\Big(\dot{K}_{ik}K_j{}^k + K_{ik}\mathcal{D}_j\mathcal{D}^k N \\
&\qquad\qquad - N\mathcal{D}_k\mathcal{D}_j K_i{}^k - N^l K_{kl}\mathcal{D}_j K_i{}^k + K_{ik}\mathcal{D}_j(N^l K_l{}^k) + \mathcal{D}^k N^l K_{ik}K_{jl}\Big).
\end{aligned} \tag{4.18}$$

After tedious calculations, we see that the expression is equal to $-2NE_{00}$, where 0 represents time direction in local Lorentz frame. This gives a consistency check that we are dealing with the correct Hamiltonian.

As mentioned before, the variations of the Hamiltonian (4.8) and (4.12) are derived up to total derivative terms. Therefore in order to derive correct equations of motion, it is necessary to add surface term $Q[\xi]$ to the Hamiltonian. By taking the two dimensional coordinates as (r, ϕ), the variation of the surface term $\delta Q[\xi]$ is given so as to cancel the total derivative terms in the variation of the Hamiltonian:

$$\begin{aligned}
&\delta Q[\xi] \\
&= \int d\phi\Big[\sqrt{g}S^{ijkr}(\xi^0\mathcal{D}_k\delta g_{ij} - \mathcal{D}_k\xi^0\delta g_{ij}) + \xi^i(2\pi^{jr} + \beta g^{\frac{1}{2}}A^{jr})\delta g_{ij} - \frac{1}{2}\xi^r(2\pi^{ij} + \beta g^{\frac{1}{2}}A^{ij})\delta g_{ij} \\
&\quad + \xi_i\delta(2\pi^{ir} + \beta g^{\frac{1}{2}}A^{ir}) - 2\beta\sqrt{g}\epsilon^{mr}\mathcal{D}_k\xi^0 g^{kl}\delta K_{ml} + 2\beta\sqrt{g}\epsilon^{mn}\xi^0\mathcal{D}_n(g^{rl}\delta K_{ml}) \\
&\quad + \frac{1}{2}\beta\sqrt{g}S^{ijkr}\big((\epsilon^{mn}\partial_m\xi_n)\mathcal{D}_k\delta g_{ij} - \mathcal{D}_k(\epsilon^{mn}\partial_m\xi_n)\delta g_{ij}\big) \\
&\quad - 2\beta\sqrt{g}\epsilon^{mr}\xi^i K_i{}^l\delta K_{ml} - \beta\sqrt{g}\epsilon^{mn}\xi^i(\delta K_{ni}K_m{}^r + K_{ni}\delta K_m{}^r) - 2\beta\sqrt{g}\epsilon^{mn}g^{rl}\xi^0 K_{mo}\delta\gamma^o{}_{nl} \\
&\quad - 2\beta\sqrt{g}\epsilon^{mn}g^{kl}g^{op}\{-\mathcal{D}_k\xi^0 K_{mo}T^{ijr}_{nlp} + \xi^0\mathcal{D}_o K_{ml}T^{ijr}_{knp} + 2\xi^0\mathcal{D}_n K_{o(l}T^{ijr}_{m)kp}\}\delta g_{ij} \\
&\quad + 2\beta\sqrt{g}\epsilon^{mn}\xi^o\{K_o{}^p K_{mk}g^{kl}T^{ijr}_{npl} + g^{qk}g^{lp}(K_{mk}K_{l(o}T^{ijr}_{n)qp} + K_{no}K_{l(k}T^{ijr}_{m)qp})\}\delta g_{ij} \\
&\quad + \frac{1}{2}\beta\sqrt{g}\epsilon^{mn}T^{xyk}_{mlo}g^{op}\{\mathcal{D}_k u_{xy}g^{lq}T^{ijr}_{npq} - u_{xy}\gamma^q{}_{np}g^{ls}T^{ijr}_{kqs} + 2u_{xy}\gamma^l{}_{q(p}g^{qs}T^{ijr}_{n)ks}\}\delta g_{ij} \\
&\quad - \frac{1}{2}\beta\sqrt{g}\epsilon^{mn}T^{ijr}_{mlo}u_{ij}g^{op}\delta\gamma^l{}_{np} + \frac{1}{2}\beta\sqrt{g}\epsilon^{mr}\xi_m\delta r\Big].
\end{aligned} \tag{4.19}$$

The index r which is not contracted represents the radial coordinate. Here we introduced $u_{ij}(\xi) \equiv 2\xi^0 K_{ij} + 2\mathcal{D}_{(i}\xi_{j)}$ and the vector $\xi = (\xi^0, \xi^r, \xi^\phi)$ is related to a Killing vector

$\bar{\xi} = (\bar{\xi}^t, \bar{\xi}^r, \bar{\xi}^\phi)$ by (3.10). Since we are dealing with the Hamiltonian, the Killing vector should be $\bar{\xi} = (1, 0, 0)$ and $\xi = (N, N^r, N^\phi)$. When we deal with the angular momentum, the Killing vector should be $\bar{\xi} = (0, 0, 1)$ and $\xi = (0, 0, 1)$. We also call ξ the Killing vector, since it does not make any confusion. The equation (4.19) makes it possible to evaluate the mass and the angular momentum of the BTZ black hole, and the central charges in TMG.

The charge $Q[\xi]$ itself is obtained by integrating (4.19) over the canonical variables with reference to the background values $\tilde{g}_{ij}$, $\tilde{\pi}_{ij}$, $\tilde{K}_{ij}$ and $(2\tilde{\pi}^{ir} + \beta\tilde{g}^{\frac{1}{2}}\tilde{A}^{ir})$. The last two become zero at the boundary. (Consult appendix B for explicit representations.) Then we are able to give an explicit form for $Q[\xi]$.

$$
\begin{aligned}
Q[\xi] = \int d\phi \Big[& \sqrt{\tilde{g}}\tilde{S}^{ijkr}\left(\xi^0\tilde{\mathcal{D}}_k(g_{ij} - \tilde{g}_{ij}) - \tilde{\mathcal{D}}_k\xi^0(g_{ij} - \tilde{g}_{ij})\right) \\
& + \xi^i(2\tilde{\pi}^{jr} + \beta\tilde{g}^{\frac{1}{2}}\tilde{A}^{jr})(g_{ij} - \tilde{g}_{ij}) - \frac{1}{2}\xi^r(2\tilde{\pi}^{ij} + \beta\tilde{g}^{\frac{1}{2}}\tilde{A}^{ij})(g_{ij} - \tilde{g}_{ij}) \\
& + \xi_i(2\pi^{ir} + \beta g^{\frac{1}{2}}A^{ir}) - 2\beta\sqrt{\tilde{g}}\tilde{\epsilon}^{mr}\tilde{\mathcal{D}}_k\xi^0\tilde{g}^{kl}K_{ml} + 2\beta\sqrt{\tilde{g}}\tilde{\epsilon}^{mn}\xi^0\tilde{\mathcal{D}}_n(\tilde{g}^{rl}K_{ml}) \\
& + \frac{1}{2}\beta\sqrt{\tilde{g}}\tilde{S}^{ijkr}\left((\tilde{\epsilon}^{mn}\partial_m\xi_n)\tilde{\mathcal{D}}_k(g_{ij} - \tilde{g}_{ij}) - \tilde{\mathcal{D}}_k(\tilde{\epsilon}^{mn}\partial_m\xi_n)(g_{ij} - \tilde{g}_{ij})\right)\Big]. \quad (4.20)
\end{aligned}
$$

The canonical variables in (ξ^0, ξ^i) should also be replaced by the background values. Thus the integrability condition for $Q[\xi]$ is satisfied and $\delta Q[\xi]$ is δ-exact. Note that in order to get this expression, we made use of $\tilde{K}_{ij} \to 0\, (r \to \infty)$ and so on, so the terms in the last five lines in eq. (4.19) are simply dropped. Note also that the integration constant in eq. (4.20) is adjusted so that the charge is zero for $(g_{ij}, \pi_{ij}, K_{ij}, 2\pi^{ir} + \beta g^{\frac{1}{2}}A^{ir}) = (\tilde{g}_{ij}, \tilde{\pi}_{ij}, 0, 0)$. Explicit calculations of several charges are done in the following subsection.

4.2 Mass and angular momentum of BTZ black hole and central charges of CFT at the boundary

First let us evaluate the mass of the BTZ black hole in TMG by using (4.19). The BTZ black hole geometry (2.6) still becomes a solution in TMG . This solution is invariant under the time translation and corresponding Killing vector ξ near the boundary is written as

$$
(\xi^0, \xi^r, \xi^\phi) = (N, N^r, N^\phi) \sim \left(\frac{r}{\ell},\ 0,\ \frac{4G_N j}{r^2}\right). \quad (4.21)
$$

In the background of BTZ black hole, two dimensional quantities which are needed to estimate the mass behave near the boundary as

$$
\delta g_{rr} \sim \frac{8G_N m\ell^4}{r^4}, \quad \delta K_{r\phi} \sim \frac{4G_N j\ell}{r^2}, \quad \delta\left(\pi^{r\phi} + \frac{1}{2}g^{\frac{1}{2}}A^{r\phi}\right) \sim \frac{4G_N j}{r^2}. \quad (4.22)
$$

It seems that there are many terms to be estimated in (4.19). Only a few terms, however, turn out to be non zero values. In fact, last six lines in (4.19) should be zero, since $\delta r \sim 0$, $\epsilon^{mn}\partial_m N_n = 0$, $K_{ij} = 0$ and $u_{ij}(\xi) = \dot{g}_{ij} = 0$. Some of remaining terms also vanish and the mass is eventually calculated as

$$
\begin{aligned}
M &= \frac{1}{16\pi G_N}\delta Q[\xi] \\
&= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ 2\sqrt{g} S^{r\phi r\phi}(-\xi^0 \gamma^r{}_{\phi\phi}\delta g_{rr}) + 2\beta \mathcal{D}_k \xi^0 g^{kl}\delta K_{\phi l} \Big\} \\
&= m + \frac{\beta}{\ell^2} j.
\end{aligned} \tag{4.23}
$$

For further details of the calculation the reader is referred to appendix B. This correctly reproduces the result obtained by the other methods [25 – 30, 8, 19]. The mass of the BTZ black hole in TMG is shifted by the angular momentum which is defined in the gravity theory with negative cosmological constant.

The angular momentum of the BTZ black hole is also calculated in a similar way. The BTZ solution is invariant under the rotation along the ϕ direction and corresponding Killing vector ξ is written as

$$
(\xi^0, \xi^r, \xi^\phi) = (0, 0, 1). \tag{4.24}
$$

Last five lines in (4.19) should be zero, since $\delta r \sim 0$, $K_{ij} = 0$ and $u_{ij}(\xi) = 2\mathcal{D}_{(i}\xi_{j)} = 0$. The evaluation of remaining terms is not so laborious and the result becomes

$$
\begin{aligned}
J &= \frac{1}{16\pi G_N}\delta Q[\xi] \\
&= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ \xi_i \delta(2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) + \beta\sqrt{g} S^{r\phi r\phi}(-\epsilon^{mn}\partial_m \xi_n \gamma^r{}_{\phi\phi}\delta g_{rr}) \Big\} \\
&= j + \beta m.
\end{aligned} \tag{4.25}
$$

Again, the angular momentum of the BTZ black hole in TMG is shifted by the mass which is defined in the gravity theory with negative cosmological constant [25 – 30, 8, 19].

Finally let us evaluate the central charge in TMG . As discussed in section 3, the diffeomorphisms which do not alter the boundary condition are labelled by $\xi_n^{\pm}$ in (3.3) whose components are given by (3.2). By using those Killing vectors, we can deform the global AdS_3 background G^0_{IJ}. Here we choose a Killing vector $\bar{\eta}$ which corresponds to one of $\xi_n^{\pm}$. Then the metric is written as

$$
G_{IJ} = G^0_{IJ} + \mathcal{D}_I \bar{\eta}_J + \mathcal{D}_J \bar{\eta}_I. \tag{4.26}
$$

From the ADM decomposition of this metric, the lapse N, the shift vector N^i and two dimensional metric g_{ij} are obtained, and further it is possible to calculate the extrinsic curvature $K_{ij} \sim -\frac{1}{2N}(\mathcal{D}_i N_j + \mathcal{D}_j N_i)$ or canonical variables such as π^{ij}. When we evaluate the central charge, we need to substitute these quantities into (4.19). Fortunately, since $\delta_\eta r$, K_{ij}

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and $u_{ij}(\xi)$ are zero at the leading order, the expression for the central charge is simplified as

$$\begin{aligned}
&\delta_\eta Q[\xi] \\
&= \int d\phi \Big[\sqrt{g} S^{ijkr} \big(\xi^0 \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k \xi^0 \delta_\eta g_{ij} \big) + \xi^i (2\pi^{jr} + \beta g^{\frac{1}{2}} A^{jr}) \delta_\eta g_{ij} \\
&\qquad - \frac{1}{2} \xi^r (2\pi^{ij} + \beta g^{\frac{1}{2}} A^{ij}) \delta_\eta g_{ij} + \xi_i \delta_\eta (2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) \\
&\qquad - 2\beta \sqrt{g} \epsilon^{mr} \mathcal{D}_k \xi^0 g^{kl} \delta_\eta K_{ml} + 2\beta \sqrt{g} \epsilon^{mn} \xi^0 \mathcal{D}_n (g^{rl} \delta_\eta K_{ml}) \\
&\qquad + \frac{1}{2} \beta \sqrt{g} S^{ijkr} \big((\epsilon^{mn} \partial_m \xi_n) \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k (\epsilon^{mn} \partial_m \xi_n) \delta_\eta g_{ij} \big) \Big] \\
&= \int d\phi \Big[\sqrt{g} S^{ijkr} \big(\xi^0 \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k \xi^0 \delta_\eta g_{ij} \big) + \xi_i \delta_\eta (2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) \\
&\qquad + 2\beta \sqrt{g} \epsilon^{rm} \partial_k \xi^0 g^{kl} \delta_\eta K_{ml} + \frac{1}{2} \beta \sqrt{g} S^{ijkr} \big((\epsilon^{mn} \partial_m \xi_n) \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k (\epsilon^{mn} \partial_m \xi_n) \delta_\eta g_{ij} \big) \Big].
\end{aligned} \tag{4.27}$$

In the above, the Killing vector ξ is constructed out of $\bar{\xi}$ as before, and its asymptotic value should be chosen out of

$$(\xi^0, \xi^r, \xi^\phi) \sim \left(\frac{r}{2} e^{inx^\pm}, -i \frac{nr}{2} e^{inx^\pm}, \pm \frac{1}{2} e^{inx^\pm} \right). \tag{4.28}$$

Second equality is derived by substituting fluctuations of canonical variables. Those are summarized in appendix B.

Now we have prepared all tools which are necessary to calculate the central charge. The central charge for $\eta = \xi_n^+$ and $\xi = \xi_m^+$ is evaluated as

$$\begin{aligned}
&\frac{1}{16\pi G_N} \delta_{\eta=\xi_n^+} Q[\xi = \xi_m^+] \\
&\quad = \frac{1}{16\pi G_N} \oint_{r=\infty} d\phi \Big\{ \frac{1}{\ell} \Big(\frac{1}{r} \xi^0 + \partial_r \xi^0 \Big) \delta_\eta g_{\phi\phi} + \frac{r^3}{\ell^3} \xi^0 \delta_\eta g_{rr} + \xi_\phi \delta_\eta (2\pi^{\phi r} + \beta g^{\frac{1}{2}} A^{\phi r}) \Big\} \\
&\quad + \frac{\beta}{16\pi G_N} \oint_{r=\infty} d\phi \left\{ \frac{1}{\ell^2} \Big(\frac{1}{r} \xi^0 + \partial_r \xi^0 \Big) \delta_\eta g_{\phi\phi} + \frac{r^3}{\ell^4} \xi^0 \delta_\eta g_{rr} + 2 \partial_r \xi^0 g^{rl} \delta_\eta K_{\phi l} \right\} \\
&\quad = -\frac{i}{12} \frac{3\ell}{2G_N} \Big(1 + \frac{\beta}{\ell} \Big) m(m^2 - 1) \delta_{m,-n}.
\end{aligned} \tag{4.29}$$

The second line just gives the contribution of Einstein-Hilbert term. The third line gives the modification. In a similar way the central charge for $\eta = \xi_n^-$ and $\xi = \xi_m^-$ is evaluated as

$$\begin{aligned}
&\frac{1}{16\pi G_N} \delta_{\eta=\xi_n^-} Q[\xi = \xi_m^-] \\
&\quad = \frac{1}{16\pi G_N} \oint_{r=\infty} d\phi \Big\{ \frac{1}{\ell} \Big(\frac{1}{r} \xi^0 + \partial_r \xi^0 \Big) \delta_\eta g_{\phi\phi} + \frac{r^3}{\ell^3} \xi^0 \delta_\eta g_{rr} + \xi_\phi \delta_\eta (2\pi^{\phi r} + \beta g^{\frac{1}{2}} A^{\phi r}) \Big\} \\
&\quad + \frac{\beta}{16\pi G_N} \oint_{r=\infty} d\phi \left\{ -\frac{1}{\ell^2} \Big(\frac{1}{r} \xi^0 + \partial_r \xi^0 \Big) \delta_\eta g_{\phi\phi} - \frac{r^3}{\ell^4} \xi^0 \delta_\eta g_{rr} + 2 \partial_r \xi^0 g^{rl} \delta_\eta K_{\phi l} \right\} \\
&\quad = -\frac{i}{12} \frac{3\ell}{2G_N} \Big(1 - \frac{\beta}{\ell} \Big) m(m^2 - 1) \delta_{m,-n}.
\end{aligned} \tag{4.30}$$

Again the central charge is given by the sum of the Einstein-Hilbert part and the gravitational Chern-Simons part. Note, however, that the signs in front of the modifications are different. From a similar calculation, it is possible to check that $\delta_{\eta=\xi_n^+} Q[\xi = \xi_m^-] = 0$. If we call ξ_m^+ left mover and ξ_m^- right one, the central charges are written as [7, 8, 18, 19]

$$\begin{aligned} c_L &= \frac{3\ell}{2G_N}\left(1 + \frac{\beta}{\ell}\right), \\ c_R &= \frac{3\ell}{2G_N}\left(1 - \frac{\beta}{\ell}\right). \end{aligned} \tag{4.31}$$

Thus via canonical formalism of topologically massive gravity, we have succeeded to realize the Virasoro algebras of left and right movers with different central charges.[3]

5. Central charges with all higher derivative corrections

5.1 Final expression of central charges

We have by now established the canonical formalism and have got the Virasoro central charges (4.31) for TMG which consists of Einstein-Hilbert action with negative cosmological constant and gravitational Chern-Simons terms. We are now in a position to generalize these results in order to encompass most general cases of higher derivative gravity. In the last paragraph of section 3, we have seen that inclusion of all higher derivative corrections other than the gravitational Chern-Simons term requires us to multiply the central charges of Brown and Henneaux's by the conformal factor Ω [10]. Making use of this simple scaling rule, we get to the following final expression of the central charges for the left and right movers:

$$\begin{aligned} c_L &= \frac{3\ell}{2G_N}\left(\Omega + \frac{\beta}{\ell}\right), \\ c_R &= \frac{3\ell}{2G_N}\left(\Omega - \frac{\beta}{\ell}\right). \end{aligned} \tag{5.1}$$

We would like to emphasize that all of the effects due to higher order terms are included in the factors Ω and β. We also note that these central charges are obtained by constructing Virasoro algebras directly in the canonical method without referring to the Wald's formula or variations thereof.

Furthermore, the definition of mass and angular momentum of BTZ black hole should necessarily be modified in the most general theory of gravity (2.1). Combining the results of (3.24), (4.23) and (4.25), the effective mass and the angular momentum become

$$M = \Omega m + \frac{\beta}{\ell^2} j, \qquad J = \Omega j + \beta m. \tag{5.2}$$

[3] See ref. [31] in the case of Riemann-Cartan geometry.

From these, the zero modes L_0^+ and L_0^- of Virasoro algebras for left and right movers are expressed as

$$L_0^+ = \frac{1}{2}(M\ell + J) = \left(\Omega + \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell + j)\,, \tag{5.3}$$

$$L_0^- = \frac{1}{2}(M\ell - J) = \left(\Omega - \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell - j)\,. \tag{5.4}$$

Putting these together with (5.1) into the Cardy's formula for counting the states in CFT, we obtain the entropy

$$\begin{aligned} S &= 2\pi\sqrt{\frac{1}{6}c_L L_0^+} + 2\pi\sqrt{\frac{1}{6}c_R L_0^-} \\ &= \frac{\pi}{2G_N}\left(\Omega + \frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m + \frac{j}{\ell}\right)} + \frac{\pi}{2G_N}\left(\Omega - \frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m - \frac{j}{\ell}\right)}. \end{aligned} \tag{5.5}$$

This agrees with the previous entropy formula (2.10) obtained by the extended Wald's formula. For the BTZ black hole capturing the contributions of all higher derivative corrections, we have thus proven the agreement between the macroscopic entropy and the Cardy's entropy of microstate counting.

5.2 Realization in M-theory: M5 system

In 5.1, we applied Brown-Henneaux's method to the three dimensional gravity theories with most general higher derivative terms, and derived the central charges of CFT at the boundary. The three dimensional theory of that sort is usually embedded in higher dimensional theories in the string theory context. Among several others, the most interesting example is embodied in M-theory, which is intriguing because the corresponding CFT is understood very clearly [5].

The M-theory is defined in eleven dimensions and its low energy limit is well described by eleven dimensional supergravity. When we compactify the eleven dimensional supergravity on CY_3, it becomes five dimensional supergravity with eight supercharges. Beyond the low energy limit, the M-theory contains a lot of terms which correct eleven dimensional supergravity. Among subleading terms in the derivative expansion, there exists a would-be Chern-Simons term which is expressed as [32]

$$\frac{\ell_p^6}{2\kappa_{11}^2}\frac{\pi^2}{3\cdot 2^6}\int A\wedge \mathrm{tr}(R\wedge R)\wedge \mathrm{tr}(R\wedge R), \tag{5.6}$$

where ℓ_p is the Planck length in eleven dimensions, $2\kappa_{11}^2 = (2\pi)^8\ell_p^9$. A is a 3-form potential and the M5-brane is magnetically coupled to this field. The Chern-Simons term in five dimensions arises after reducing this term. In fact, by expanding the 3-form with a basis of harmonic $(1,1)$-forms $J_{\hat{I}}$ in CY_3 as $A = 8\pi^2\ell_p^3 A^{\hat{I}}\wedge J_{\hat{I}}$, we obtain [33]

$$\frac{c_{2\hat{I}}}{3\cdot 2^7\pi^2}\int A^{\hat{I}}\wedge \mathrm{tr}(R\wedge R). \tag{5.7}$$

Here $A^{\hat{I}}$ corresponds to gauge fields in five dimensions, and $c_{2\hat{I}} = \frac{1}{8\pi^2}\int_{\mathrm{CY}_3} J_{\hat{I}} \wedge \mathrm{tr}(R\wedge R)$. In general, other curvature squared terms are also obtained by the dimensional reduction of R^4 terms in the M-theory. It is, however, complicated to work out all these terms, thus we employ the five dimensional conformal supergravity below.

By applying the conformal supergravity approach, the action of the five dimensional supergravity theory concerned with R^2 terms has been constructed in ref. [34]. With this action together with an ansatz of $\mathrm{AdS}_3 \times \mathrm{S}^2$ geometry, we would like to identify the effective cosmological constant $-2/\ell^2$, the conformal factor Ω and the coupling constant β in terms of topological quantities in CY_3. Actually the three dimensional gravitational Chern-Simons term can be obtained by compactifying (5.7) on S^2. The central charges are also expressed by the CY_3 data in the context of string theory. In passing note that similar calculations have been done in refs. [7, 8, 12].

In the following we employ notations used in ref. [12]. Assuming that the five dimensional metric, gauge fields and 2-form auxiliary field v be given by

$$\begin{aligned} ds^2_{(5)} &= \psi^2 G_{IJ}dx^I dx^J + \chi^2 d\Omega^2_{S^2}, \\ F^{\hat{I}}_{\theta\phi} &= \frac{p^{\hat{I}}}{2}\sin\theta, \\ v_{\theta\phi} &= V\sin\theta, \end{aligned} \tag{5.8}$$

we obtain the three dimensional supergravity with the curvature squared terms and gravitational Chern-Simons term. Here $p^{\hat{I}}$ corresponds to the M5-brane charge. In order to realize the Einstein frame in three dimensions, we have to set

$$\psi^{-1} = \frac{\chi^2}{\pi}\left(\frac{3}{4} + \frac{1}{4}\mathcal{N} + \frac{c_{2\hat{I}}M^{\hat{I}}}{288\chi^2} + \frac{c_{2\hat{I}}M^{\hat{I}}V^2}{72\chi^4} - \frac{c_{2\hat{I}}p^{\hat{I}}V}{288\chi^4}\right), \tag{5.9}$$

where $M^{\hat{I}}$ stand for moduli scalars. Then the action becomes [12]

$$\mathcal{S} = \int d^3x\sqrt{-G}\left(R + Z(\phi) + A(\phi)R^2 + B(\phi)R_{IJ}R^{IJ}\right) + \int d^3x\mathcal{L}_{\mathrm{CS}} + \mathcal{S}', \tag{5.10}$$

in the unit of $16\pi G_N = 1$. Here ϕ stands generically for all scalars $M^{\hat{I}}$, V, D and χ, and $\mathcal{S}'$ includes their derivative terms. The scalar potential and each coupling are given by

$$\begin{aligned} Z(\phi) = \psi^3\frac{\chi^2}{\pi}\Bigg\{ &\frac{2}{\chi^2}\left(\frac{3}{4}+\frac{\mathcal{N}}{4}\right) - 2\left(\frac{D}{4} - \frac{V^2}{\chi^4}\right) + \mathcal{N}\left(\frac{D}{2} + \frac{6V^2}{\chi^4}\right) + \frac{2\mathcal{N}_{\hat{I}}p^{\hat{I}}V}{\chi^4} + \frac{\mathcal{N}_{\hat{I}\hat{J}}p^{\hat{I}}p^{\hat{J}}}{8\chi^4} \\ &+ \frac{c_{2\hat{I}}M^{\hat{I}}}{96\chi^4} + \frac{c_{2\hat{I}}M^{\hat{I}}D^2}{288} + \frac{c_{2\hat{I}}p^{\hat{I}}VD}{144\chi^4} - \frac{5c_{2\hat{I}}M^{\hat{I}}V^2}{36\chi^6} - \frac{c_{2\hat{I}}p^{\hat{I}}V}{48\chi^6} + \frac{c_{2\hat{I}}p^{\hat{I}}V^3}{36\chi^8} + \frac{c_{2\hat{I}}M^{\hat{I}}V^4}{6\chi^8}\Bigg\}, \\ A(\phi) = -\frac{5}{6}&\frac{c_{2\hat{I}}M^{\hat{I}}\chi^2}{192\pi\psi} \quad , \quad B(\phi) = \frac{8}{3}\frac{c_{2\hat{I}}M^{\hat{I}}\chi^2}{192\pi\psi} \quad , \quad \beta = -\frac{c_{2\hat{I}}p^{\hat{I}}}{96\pi}, \end{aligned} \tag{5.11}$$

where

$$\mathcal{N} = \frac{1}{6}c_{\hat{I}\hat{J}\hat{K}}M^{\hat{I}}M^{\hat{J}}M^{\hat{K}} \quad , \quad \mathcal{N}_{\hat{I}} = \frac{1}{2}c_{\hat{I}\hat{J}\hat{K}}M^{\hat{J}}M^{\hat{K}} \quad , \quad \mathcal{N}_{\hat{I}\hat{J}} = c_{\hat{I}\hat{J}\hat{K}}M^{\hat{K}}. \tag{5.12}$$

$c_{\hat{I}\hat{J}\hat{K}}$ and $c_{2\hat{I}}$ are the triple intersection number and the second Chern class number of CY_3, respectively.

The action (5.10) enables us to derive an equation of motion for the metric

$$\frac{1}{2}G^{IJ}\{R+AR^2+B(R_{IJ})^2+Z\}-R^{IJ}-2ARR^{IJ}-2BR^{IK}R_K^J+T^{IJ}$$
$$=\beta\epsilon^{KL(I}\mathcal{D}_K R_L^{J)}+(\text{derivative terms of }\phi),\qquad(5.13)$$

and those for scalars

$$\partial_\phi Z+\partial_\phi AR^2+\partial_\phi B(R_{IJ})^2=(\text{derivative terms of }\phi).\qquad(5.14)$$

It is, however, almost impossible for us to find general solutions to these equations. What we can do is to take all ϕ to be constants everywhere and to assume that the BTZ black hole which satisfies (2.4). It corresponds to the black ring solution whose geometry is $\mathrm{AdS}_3\times\mathrm{S}^2$ in five dimension. By substituting (2.4), equations of motion (5.13) and (5.14) reduce to

$$\begin{aligned}Z&=\frac{2}{\ell^2}+(3A+B)\left(\frac{2}{\ell^2}\right)^2,\\0&=\partial_\phi Z+3\,(3\,\partial_\phi A+\partial_\phi B)\left(\frac{2}{\ell^2}\right)^2.\end{aligned}\qquad(5.15)$$

Since $c_{2\hat{I}}$ indicates the higher derivative corrections in the next order, we can solve five equations (5.15) to the first order of $c_{2\hat{I}}$. The solutions are

$$M^{\hat{I}}=\frac{p^{\hat{I}}}{p}\left(1-\frac{C}{36}\right),\ V=-\frac{3}{8}p\left(1+\frac{C}{36}\right),\ D=\frac{12}{p^2}\left(1-\frac{C}{18}\right),\ \chi=\frac{p}{2}\left(1+\frac{C}{36}\right),\qquad(5.16)$$

and

$$\ell=\frac{p^3}{4\pi}\left(1+\frac{37}{288}C\right),\qquad(5.17)$$

where $p^3=\frac{1}{6}c_{\hat{I}\hat{J}\hat{K}}p^{\hat{I}}p^{\hat{J}}p^{\hat{K}}$ and $C=c_{2\hat{I}}p^{\hat{I}}/p^3$. On the other hand, the conformal factor Ω for this solution is calculated as

$$\begin{aligned}\Omega(\ell)&=1+2AR+\frac{2}{3}BR\\&\simeq 1-\frac{C}{288}.\end{aligned}\qquad(5.18)$$

The assumption of constant scalars admits the BTZ black hole solution. Therefore, the Brown-Henneaux's approach explained in the previous sections can be applied to this solution, and we can prove the existence of the two dimensional CFT satisfying the Virasoro algebra on the AdS boundary. With the use of $16\pi G_N=1$, $\beta=-c_{2\hat{I}}p^{\hat{I}}/96\pi$ and the formula (5.1), the central charges of the left and right movers are given by

$$\begin{aligned}c_L&=6p^3+\frac{1}{2}c_{2\hat{I}}p^{\hat{I}},\\c_R&=6p^3+c_{2\hat{I}}p^{\hat{I}},\end{aligned}\qquad(5.19)$$

in agreement with [5, 7, 8, 35, 12]. In four dimensions, these central charges appear in expressions of the entropy for the extremal non-BPS and BPS black holes, respectively [36]. The precise information of the microstates for the CFT at the boundary is veiled in our formalism. Whatever the microstates may be, we can only see the Virasoro algebras and calculate their central charges. But as for the M5-brane system, the explanation for microstates was made clear in ref. [5] from the detailed description of the effective field theory on the brane.

6. Summary and discussions

In the present paper we have analyzed topologically massive gravity (1.1), using the conventional canonical formalism. Since there are higher derivative terms w.r.t. time, we made use of the generalized version of the Ostrogradsky method. We defined the global charges so as to cancel the surface terms of the variation of the Hamiltonian. Using these, we have derived the Virasoro algebras realized asymptotically at the boundary ($r \sim \infty$), and found that the central charges are given by (4.31), which do not respect left-right symmetry. The mass and the angular momentum of the BTZ black hole are also computed including the effects due to the gravitational Chern-Simons term.

We have gone one step further to argue that effects due to higher derivative terms can be included by employing the scaling argument. The central charges, the mass and the angular momentum in such a general class of higher derivative theories are given by (5.1) and (5.2), respectively. The BTZ black hole entropy is given by (5.5), which agrees with the formula given by using the modified Wald's formula. We have thus succeeded in strengthening the link between the two dimensional boundary CFT and the three dimensional gravity description of the black hole. As an interesting example, we considered the three dimensional model which are realized by compactifying the M-theory on $\mathrm{CY}_3 \times \mathrm{S}^2$. The left-right asymmetric central charges are already given in ref. [5] from the microscopic viewpoint, and we confirmed that the result can exactly be reproduced in our formalism. The consideration from the representation of Virasoro algebras is also important as future works [37].

In this paper, we treated ℓ and β as free parameters. From the analyses by CFT, however, it was pointed out that the three dimensional gravity theory could be realized so as to be consistent with unitarity and positivity only when ℓ and β take some particular values [38, 39].[4] It would be interesting if we could re-examine these observations from our canonical approach including the higher derivative contribution Ω. Generalization of our formalism to the supergravity is also an interesting future direction [40].

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[4]For recent arguments, see refs. [41 – 43].

A. ADM decomposition of gravitational Chern-Simons term

In this appendix we give a quick summary of the ADM decomposition of the gravitational Chern-Simons term. First note that capital variables G_{IJ}, $E^A_{\ I}$, $\Gamma^I_{\ JK}$ and $\Omega^A_{\ BI}$ are three dimensional metric, vielbein, affine connection and spin connection, respectively, and g_{ij}, $e^a_{\ i}$, $\gamma^i_{\ jk}$ and $\omega^a_{\ bi}$ are two dimensional ones. As in the main body of the text, $I, J \cdots = t, r, \phi$ labels the three dimensional space-time indices and $A, B \cdots = 0, 1, 2$ denotes the three dimensional local Lorentz indices. Also, small indices are of two dimensional: $i, j \cdots = r, \phi$ and $a, b \cdots = 1, 2$.

For the purpose of the ADM decomposion of the gravitational Chern-Simons term (1.3), the simplest way is to divide $\mathcal{L}_{\rm CS}$ into the Lorentz Chern-Simons and remaining terms. When we write the connection 1-form expressed by the matrix notation as $\Gamma^I_{\ J} = \Gamma^I_{\ JK} dx^K$, the relation between the affine connection and the spin connection is

$$\Gamma^I_{\ J} = E^I_{\ A} \Omega^A_{\ B} E^B_{\ J} + E^I_{\ A} dE^A_{\ J}. \tag{A.1}$$

Note that $\Gamma^I_{\ J}$ and $\Omega^A_{\ B}$ are the connection 1-form but $E^A_{\ J}$ is the 0-form vielbein. Omitting the indices like $\Gamma = E^{-1}\Omega E + E^{-1} dE$, the gravitational Chern-Simons term is expressed by

$$\mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) = \mathrm{Tr}\left(\Omega d\Omega + \frac{2}{3}\Omega^3\right) - \frac{1}{3}\mathrm{Tr}(dEE^{-1})^3 - d\,\mathrm{Tr}(dEE^{-1}\Omega). \tag{A.2}$$

We can drop the last term since it is just a total derivative.

Let us define

$$K_{ab} = \frac{1}{N}(e^i_{\ (a}\dot{e}_{b)\,i} - D_{(a}N_{b)}) \quad , \quad L_{ab} = \frac{1}{N}(e^i_{\ [a}\dot{e}_{b]\,i} - D_{[a}N_{b]}), \tag{A.3}$$

where D_i is the covariant derivative which acts on the local Lorentz indices like $D_i V^a = \partial_i V^a + \omega^a_{\ bi} V^b$. Then the first term of (A.2) becomes

$$\begin{aligned} \Omega^A_{\ B} d\Omega^B_{\ A} &= 2\Omega^0_{\ a} d\Omega^a_{\ 0} + \Omega^a_{\ b} d\Omega^b_{\ a} \\ &= 2\{-(\partial_a N + K_{ab}N^b)\partial_j K^a_{\ i} + \partial_j(\partial^a N + K^a_{\ b}N^b)K_{ai} + K_{aj}\dot{K}^a_{\ i}\} dt\wedge dx^i \wedge dx^j \\ &\quad + \{\omega^a_{\ bi}\partial_j(L^b_{\ a}N) - \partial_j\omega^b_{\ ai}L^a_{\ b}N - \omega^a_{\ bi}\dot{\omega}^b_{\ aj}\} dt\wedge dx^i\wedge dx^j, \end{aligned} \tag{A.4}$$

and the second term is

$$\begin{aligned} \Omega^A_{\ B}\Omega^B_{\ C}\Omega^C_{\ A} &= 3\Omega^0_{\ a}\Omega^a_{\ b}\Omega^b_{\ 0} \\ &= 3(2\partial_a N\omega^a_{\ bi}K^b_{\ j} - NK_{ai}K^b_{\ j}L^a_{\ b} + K_{aj}K^b_{\ i}\omega^a_{\ bc}N^c \\ &\quad + 2K_{ae}K^b_{\ j}\omega^a_{\ bi}N^e) dt\wedge dx^i\wedge dx^j. \end{aligned} \tag{A.5}$$

If we use $\epsilon^{tij}\sqrt{-G} = \epsilon^{ij}\sqrt{g}$ and the two dimensional identity $r^{ab}_{\ \ ij} = re^{[a}_{\ [i}e^{b]}_{\ j]}$, the Lorentz Chern-Simons term can be written as

$$\begin{aligned} \mathrm{Tr}\left(\Omega d\Omega + \frac{2}{3}\Omega^3\right) \cong \{&4(\partial_a N + K_{ab}N^b)D_i K^a_{\ j} + 2K_{aj}\dot{K}^a_{\ i} - \omega^a_{\ bi}\dot{\omega}^b_{\ aj} \\ &+ (2K_{aj}K^b_{\ i} + re^b_{\ i}e_{aj})(D_b N^a - e^k_{\ b}\dot{e}^a_{\ k})\}\sqrt{g}\epsilon^{ij} d^3x \end{aligned}$$

$$
\begin{aligned}
&= \{4(\partial_l N + K_{lk}N^k)\mathcal{D}_i K^l{}_j + 2\dot{K}^k{}_i K_{kj} + 2K_{kj}K^l{}_i \mathcal{D}_l N^k + r\mathcal{D}_i N_j \\
&\qquad - re_{aj}\dot{e}^a{}_i - \omega^a{}_{bi}\dot{\omega}^b{}_{aj}\}\sqrt{g}\epsilon^{ij}d^3x,
\end{aligned} \tag{A.6}
$$

up to total derivative. K_{ij} is of course the extrinsic curvature (3.7) and $\mathcal{D}_i$ is the usual covariant derivative.

On the other hand, one can check that the second term on the right hand side of (A.2) becomes

$$
\begin{aligned}
-\frac{1}{3}\mathrm{Tr}(dEE^{-1})^3 &= -\frac{1}{3}(dEE^{-1})^a{}_b(dEE^{-1})^b{}_c(dEE^{-1})^c{}_a \\
&= \dot{e}^k{}_b \partial_i e^b{}_l e^l{}_c \partial_j e^c{}_k dt \wedge dx^i \wedge dx^j.
\end{aligned} \tag{A.7}
$$

After some manipulation and neglecting the total derivative, the sum of the last lines in (A.6) and (A.7) becomes

$$
\begin{aligned}
&\int d^3x\sqrt{g}\epsilon^{ij}\left(-re_{aj}\dot{e}^a{}_i - \omega^a{}_{bi}\dot{\omega}^b{}_{aj} + \dot{e}^k{}_b\partial_i e^b{}_l e^l{}_c\partial_j e^c{}_k\right) \\
&\qquad = \int d^3x\sqrt{g}\epsilon^{ij}\dot{\gamma}^l{}_{ik}\gamma^k{}_{jl} \\
&\qquad = \int d^3x\sqrt{g}(-\epsilon^{mp}g^{lo}T^{ijk}_{mno}\mathcal{D}_k\gamma^n{}_{pl})\dot{g}_{ij},
\end{aligned} \tag{A.8}
$$

in which we used the definition (4.3) of T^{ijk}_{mno} and

$$
\mathcal{D}_k\gamma^n{}_{pl} \equiv \partial_k\gamma^n{}_{pl} + \gamma^n{}_{km}\gamma^m{}_{pl} - \gamma^m{}_{kp}\gamma^n{}_{ml} - \gamma^m{}_{kl}\gamma^n{}_{pm}. \tag{A.9}
$$

Defining A^{ij} by (4.4), we can rewrite the last line of (A.8) as

$$
\begin{aligned}
-\int d^3x\sqrt{g}A^{ij}\dot{g}_{ij} &= -\int d^3x\sqrt{g}A^{ij}(2NK_{ij} + 2\mathcal{D}_iN_j) \\
&\cong \int d^3x\sqrt{g}(-2A^{ij}NK_{ij} + 2\mathcal{D}_iA^{ij}N_j).
\end{aligned} \tag{A.10}
$$

In fact, the result of the central charge for the Virasoro algebra does not depend on whether we use the gravitational Chern-Simons or Lorentz Chern-Simons. As we have seen in section 4, this is due to the fact that the above terms involving A^{ij} have no contribution to the central charge calculation. To sum up, combination of all the terms calculated above leads us finally to the following expression:

$$
\begin{aligned}
&\sqrt{-G}\epsilon^{IJK}\left(\Gamma^P{}_{IQ}\partial_J\Gamma^Q{}_{KP} + \frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\right) \\
&= \sqrt{g}\Big[2\epsilon^{mn}\dot{K}_{mk}K_n{}^k + N\{4\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k - 2A^{kl}K_{kl}\} \\
&\qquad + N^i\{-4\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml} - 2\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) + g_{im}\epsilon^{mn}\partial_n r + 2\mathcal{D}_kA_i{}^k\}\Big].
\end{aligned} \tag{A.11}
$$

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B. Supplementary calculations on charges

B.1 Mass and angular momentum of BTZ black hole

Let us consider the ADM decomposition of the BTZ black hole solution (2.6). From the canonical procedure, the lapse, the shift vector and the two dimensional metric are given by

$$\begin{aligned} N &= \tilde{N} + \delta N \sim \frac{r}{\ell} - \frac{4G_N m\ell}{r}, \\ N^r &= \tilde{N}^r + \delta N^r \sim 0 + 0, \\ N^\phi &= \tilde{N}^\phi + \delta N^\phi \sim 0 + \frac{4G_N j}{r^2}, \\ g_{ij} &= \tilde{g}_{ij} + \delta g_{ij} \sim \begin{pmatrix} \frac{\ell^2}{r^2} & 0 \\ 0 & r^2 \end{pmatrix} + \begin{pmatrix} \frac{8G_N m\ell^4}{r^4} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.1}$$

These are expanded around $m = j = 0$ and the flucutuations are linearly dependent on m or j. From the metric $\tilde{g}_{ij}$ given in the above, nonzero components of the affine connection and $\tilde{S}^{ijkl} = \frac{1}{2}(\tilde{g}^{ik}\tilde{g}^{jl} + \tilde{g}^{il}\tilde{g}^{jk} - 2\tilde{g}^{ij}\tilde{g}^{kl})$ are evaluated as

$$\begin{aligned} &\tilde{\gamma}^r{}_{rr} = -\frac{1}{r}, \qquad &&\tilde{\gamma}^\phi{}_{r\phi} = \frac{1}{r}, \qquad &&\tilde{\gamma}^r{}_{\phi\phi} = -\frac{r^3}{\ell^2}, \\ &\tilde{S}^{\phi\phi rr} = -\frac{1}{\ell^2}, \qquad &&\tilde{S}^{\phi r\phi r} = \frac{1}{2\ell^2}. \end{aligned} \tag{B.2}$$

By using the equations of motion (4.13) and (4.15), the extrinsic curvature and the covariant combination of the momentum are calculated as

$$\begin{aligned} K_{ij} &= \tilde{K}_{ij} + \delta K_{ij} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{4G_N j\ell}{r^2} \\ \frac{4G_N j\ell}{r^2} & 0 \end{pmatrix}, \\ \left(\pi + \frac{1}{2}\beta g^{\frac{1}{2}} A\right)^{ij} &= \left(\tilde{\pi} + \frac{1}{2}\beta \tilde{g}^{\frac{1}{2}} \tilde{A}\right)^{ij} + \delta\left(\pi + \frac{1}{2}\beta g^{\frac{1}{2}} A\right)^{ij} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{4G_N j}{r^2} \\ \frac{4G_N j}{r^2} & 0 \end{pmatrix}. \end{aligned} \tag{B.3}$$

In order to derive these expressions, we dropped terms with time derivatives.

B.2 The central charges

Let us consider the geometry constructed by $G_{IJ} = G^0_{IJ} + \mathcal{D}_I\bar{\eta}_J + \mathcal{D}_J\bar{\eta}_I$. Here G^0_{IJ} is the metric of global AdS_3 and $\bar{\eta}_I$ represents some Killing vector. For $\eta = \xi^\pm_n$, the lapse, the shift vector and the two dimensional metric behave asymptotically as

$$\begin{aligned} N &= \tilde{N} + \delta N \sim \left(\frac{r}{\ell} + \frac{\ell}{2r}\right) - \frac{i\ell n(n^2-2)}{4r} e^{inx^\pm}, \\ N^r &= \tilde{N}^r + \delta N^r \sim 0 - \frac{\ell n^2}{r} e^{inx^\pm}, \end{aligned}$$

$$N^{\phi} = \tilde{N}^{\phi} + \delta N^{\phi} \sim 0 \pm \frac{i\ell n(n^2-1)}{2r^2} e^{inx^{\pm}}, \tag{B.4}$$
$$g_{ij} = \tilde{g}_{ij} + \delta_\eta g_{ij} \sim \begin{pmatrix} \frac{\ell^2}{r^2} - \frac{\ell^4}{r^4} & 0 \\ 0 & r^2 \end{pmatrix} + \begin{pmatrix} -\frac{in\ell^4}{r^4} & \mp\frac{n^2\ell^4}{2r^3} \\ \mp\frac{n^2\ell^4}{2r^3} & \frac{in^3\ell^2}{2} \end{pmatrix} e^{inx^{\pm}}.$$

From the metric $\tilde{g}_{ij}$ given in the above, asymptotic behaviors of the affine connection and $\tilde{S}^{ijkl}$ are evaluated as (B.2). By using the equations of motion (4.13) and (4.15), the extrinsic curvature and the covariant combination of the momentum are calculated as

$$K_{ij} = \tilde{K}_{ij} + \delta_\eta K_{ij} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{2n^2\ell^4}{r^5} & \pm\frac{in(n^2-1)\ell^2}{2r^2} \\ \pm\frac{in(n^2-1)\ell^2}{2r^2} & \frac{n^2(n^2+1)\ell^2}{2r} \end{pmatrix} e^{inx^{\pm}},$$
$$\begin{aligned} \left(\pi + \frac{1}{2}\beta g^{\frac{1}{2}} A\right)^{ij} &= \left(\tilde{\pi} + \frac{1}{2}\beta \tilde{g}^{\frac{1}{2}} \tilde{A}\right)^{ij} + \delta_\eta \left(\pi + \frac{1}{2}\beta g^{\frac{1}{2}} A\right)^{ij} \\ &\sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{n^2(\ell \mp \beta + n^2(\ell \pm \beta))}{2r} & \pm\frac{in(n^2-1)\ell}{2r^2} \\ \pm\frac{in(n^2-1)\ell}{2r^2} & \pm\frac{n^2(\pm 4\ell + \beta(n^2-1))\ell^2}{2r^5} \end{pmatrix} e^{inx^{\pm}}. \end{aligned} \tag{B.5}$$

It is useful to note that $\tilde{\mathcal{D}}_n(\tilde{g}^{rl}\delta_\eta K_{ml})$ is symmetric under the exchange of m and n, i.e.,

$$\tilde{\mathcal{D}}_n(\tilde{g}^{rl}\delta_\eta K_{ml}) = \begin{pmatrix} \frac{6n^2\ell^2}{r^4} & \mp\frac{in(n^2-1)}{r} \\ \mp\frac{in(n^2-1)}{r} & -n^2(n^2+2) \end{pmatrix} e^{inx^{\pm}}. \tag{B.6}$$

Other useful equations used in the text are:

$$\tilde{g}^{rl}\delta_\eta K_{\phi l} \sim \pm i\frac{1}{2}n(n^2-1)e^{inx^{\pm}}, \quad \epsilon^{pq}\partial_p \xi_q \sim \pm\frac{r}{\ell}e^{inx^{\pm}} \sim \pm\frac{2}{\ell}\xi^0. \tag{B.7}$$

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